Special Case and Statistical Properties of Mittag-Leffler Density

حالة خاصة لدالة كثافة ميتاج ليفلر وبعض خصائصها

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Abstract

In this paper, we discussed some statistical Properties of Mittag-Leffler density with $\alpha = 1$. We used some properties of gamma density function, Laplace transform to obtain the moment generating function of Mittag-Leffler density for the sum of two distributed random variables.

المستخلص

في هذه الورقة ناقشنا بعض الخصائص الإحصائية لدالة كثافة ميتاج ليفلر عند $\alpha = 1$. إستخدمنا بعض خصائص دالة كثافة قاما. إستخدمنا تحويل لابلاس للحصول على الدالة المولدة للعزوم لمجموع متغيرين عشوائيين يتبعان توزيع دالة كثافة ميتاج ليفلر .

Keywords: Mittag-Leffler Functions, Laplace Transform, Gamma Function, Moment Generating Function.

1. Introduction:

The Mittag-Leffler function is a generalization of the exponential series was introduced by Magnus Gustaf. The Mittag-Leffler density was introduced by many researcher, for example, R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogosin see [2] , Seema S. Nair see [3] . Mathai see [4] has introduced a general statistical density related to a 3-parameter Mittag-Leffler function and the following discussion depends on modified methods that was introduced by Mathai and Haubold see [1] . This Mittag-Leffler density has important properties which many researches discussed.

2 . Mittag-Leffler Statistical Density. [1]

Mittag-Leffler function is denoted and defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \qquad Real(\alpha) > 0 (2.1)$$

And $E_\alpha(z)$ is a generalization of the series of exponential functions. The following Mittag-Leffler function is generalization of $E_\alpha(z)$ called 2-parameter Mittag-Leffler function, it is denoted and defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, \quad Real(\alpha) > 0\ , \ Real(\beta) > 0 \quad (2.2)$$

The most important generalization of $E_\alpha(z)$ called 3-parameter Mittag-Leffler function, it is denoted and defined as

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!\ \Gamma(\beta + \alpha k)}, \quad Real(\alpha) > 0\ , Real(\beta) > 0 \quad (2.3)$$

Where

$$(\gamma)_k = \gamma(\gamma + 1)(\gamma + 2) \dots (\gamma + k - 1) \quad (2.4)$$

And $(\gamma)_k$ is called Pochhmmer symbol

According to Gorenflo[2] , the Mittag-Leffler function is the most important function in the fractional calculus field. Consider a density of the following form

$$f(x) = \frac{x^{\alpha\beta-1}}{\delta^\beta} \sum_{k=0}^{\infty} \frac{(\beta)_k}{k!} \frac{(-1)^k (x^\alpha)^k}{\delta^k\ \Gamma(\alpha\beta + \alpha k)} \quad (2.5)$$

For $Real(\alpha) > 0\ ,\ Real(\beta) > 0\ ,\ x \geq 0\ ,\ \delta > 0$

Let us rewrite (2.5) without Pochhammer symbol $(\beta)_k$, and substitute

$(\beta)_k = \frac{\Gamma(\beta+k)}{\Gamma(\beta)}$ in (2.5) , we get

$$f(x) = \frac{x^{\alpha\beta-1}}{\delta^\beta \Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\beta + k)}{k!} \frac{(-1)^k (x^\alpha)^k}{\delta^k\ \Gamma(\alpha\beta + \alpha k)} \quad (2.6)$$

For $Real(\alpha) > 0\ ,\ Real(\beta) > 0\ ,\ x \geq 0\ ,\ \delta > 0$

3. Properties of the Mittag-Leffler Density.

Let us find the moment generating function $f(t)$ for Mittag-Leffler density, with parameter t, by taking the expected value of the exponential function e^{tx} , we get

$$f(t) = E(e^{tx}) = \int_0^\infty f(x)\, e^{tx}\, dx$$

$$= \frac{1}{\delta^\beta \Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\beta + k)}{k!} \frac{(-1)^k}{\delta^k\ \Gamma(\alpha\beta + \alpha k)} \int_0^\infty x^{\alpha\beta+\beta k-1} e^{tx}\, dx \quad (3.1)$$

For $t \leq 0$

Put $tx = -y$ in the integration in (3.1), we get gamma function [10] for $t \leq 0$ and (3.1) become

$$f(t) =$$

$$\frac{1}{\delta^{\beta}\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)}{k!}\frac{(-1)^k}{\delta^k\,\Gamma(\alpha\beta+\alpha k)}\,\Gamma(\alpha\beta+\alpha k)(-t)^{-(\alpha\beta+\alpha k)}$$

$$=\frac{1}{\delta^{\beta}\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)}{k!}\frac{(-1)^k}{\delta^k}(-t)^{-(\alpha\beta+\alpha k)} \qquad (3.2)$$

By using binomial theorem, we get

$$f(t) = (1+\delta(-t)^{\alpha})^{-\beta} \qquad (3.3)$$

Let us substitute$t = 0$, we get

$$f(0) = E(e^0) = E(1) = \int_0^{\infty} f(x)\ dx = (1+0)^{-\beta} = 1 \qquad (3.4)$$

That means $f(x)$ is a density function.

4. Special Case α= 1 .

Let us substitute $\alpha = 1$ in (2.6), then $f(x)$ written as

$$f(x) = \frac{x^{\beta-1}}{\delta^{\beta}\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)}{k!}\frac{(-1)^k(x)^k}{\delta^k\,\Gamma(\beta+k)}$$

$$=\frac{x^{\beta-1}}{\delta^{\beta}\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-x}{\delta}\right)^k \quad (4.1)$$

From expansion of exponential function, we get

$$f(x) = \frac{x^{\beta-1}}{\delta^{\beta}\Gamma(\beta)}e^{\frac{-x}{\delta}} \qquad (4.2)$$

This is the gamma density function, with the parameters β and $\frac{1}{\delta}$ [5]

4.1. The mean μ of the Mittag-Leffler density at α= 1 .

In this case the mean of the Mittag-Leffler density is the mean of gamma density, it is the expected value of x

$$\mu = E(x) = \int_0^{\infty} x\, f(x)\ dx$$

$$=\int_0^{\infty} x\ \frac{x^{\beta-1}}{\delta^{\beta}\Gamma(\beta)}e^{\frac{-x}{\delta}}\ dx$$

$$=\frac{1}{\delta^{\beta}\Gamma(\beta)}\int_0^{\infty} x^{\beta}e^{\frac{-x}{\delta}}\ dx \quad (4.3)$$

From gamma function (4.3) become

$$\mu = \frac{1}{\delta^{\beta}\Gamma(\beta)}\,\Gamma(\beta+1)\delta^{\beta+1} = \frac{1}{\delta^{\beta}\Gamma(\beta)}\,\beta\,\Gamma(\beta)\delta^{\beta+1} = \beta\delta \quad (4.4)$$

Another method:

From (3.3), the moment generating function at $\alpha = 1$ is

$$f(t) = (1-\delta t)^{-\beta} \quad (4.5)$$

We can find the first derivative of moment generating function $f(t)$[9], we get

$$\frac{d}{dt}(1-\delta t)^{-\beta} = \beta\delta\,(1-\delta t)^{-\beta-1} \quad (4.6)$$

And substitute$t = 0$, we get

$$f'(0) = \mu = E(x) = \beta\delta\,(1-0)^{-\beta-1} = \beta\delta \quad (4.7)$$

4.2. The variance σ^2 of the Mittag-Leffler density at α= 1 .

In this case the variance of the Mittag-Leffler density is the variance of gamma density, and we can find it by taking the expected value of x and x^2

$$E(x^2) = \int_0^{\infty} x^2\, f(x)\ dx$$

$$= \int_0^{\infty} x^2 \frac{x^{\beta-1}}{\delta^{\beta}\Gamma(\beta)} e^{\frac{-x}{\delta}}\ dx$$

$$= \frac{1}{\delta^{\beta}\Gamma(\beta)} \int_0^{\infty} x^{\beta+1} e^{\frac{-x}{\delta}}\ dx \quad (4.8)$$

From gamma function (4.8) become

$$E(x^2) = \frac{1}{\delta^{\beta}\Gamma(\beta)}\,\Gamma(\beta+2)\delta^{\beta+2} = \frac{1}{\delta^{\beta}\Gamma(\beta)}\,\beta(\beta+1)\Gamma(\beta)\delta^{\beta+2}$$

$$= \beta(\beta+1)\delta^2 \quad (4.9)$$

We know that

$$\sigma^2 = E(x^2) - [E(x)]^2 \quad (4.10)$$

From equations (4.4) and (4.9), (4.10) become

$$\sigma^2 = \beta(\beta+1)\delta^2 - (\beta\delta)^2 = \beta\delta^2 \quad (4.11)$$

Another method:

We can find the second derivative of moment generating function$f(t)$, we get

$$\frac{d^2}{dt^2}(1-\delta t)^{-\beta} = \beta(\beta+1)\delta^2(1-\delta t)^{-\beta-2} \quad (4.12)$$

And substitute $t = 0$, we get

$$f''(0) = E(x^2) = \beta(\beta+1)\delta^2(1-0)^{-\beta-1} = \beta(\beta+1)\delta^2 \qquad (4.13)$$

Substitute (4.7) and (4.13) in (4.10), we get (4.11)

4.3. The density of a sum.

Given $u > 0$ and $v > 0$ are two independently distributed random variables with the density functions $f(u)$ and $g(v)$respectively. Let $x = u + v$. Then the density of x is

$$h(x) = \int_0^x f(x-t)\, g(t)\, dt \qquad (4.14)$$

If $v > 0$ is a distributed random variable with Mittag-Leffler density at $\alpha = 1$, and $u > 0$ is a distributed random variable with another density function$f(u)$. Then the density of $x = u + v$ is

$$h(x) = \int_0^x f(x-t)\frac{t^{\beta-1}}{\delta^\beta \Gamma(\beta)} e^{\frac{-t}{\delta}}\, dt \qquad (4.15)$$

And its moment generating function with the parameter $-s$ is equal to

$$g(-s) = \int_0^\infty \left(\int_0^x f(x-t)\frac{t^{\beta-1}}{\delta^\beta \Gamma(\beta)} e^{\frac{-t}{\delta}}\, dt \right) e^{-sx}\, dx \qquad (4.16)$$

Where$s \geq 0$

From convolution theorem [10], (4.16) become

$$g(-s) = \frac{1}{\delta^\beta \Gamma(\beta)} L\{f(t)\}\, L\left\{t^{\beta-1} e^{\frac{-t}{\delta}}\right\} \qquad (4.17)$$

Where$L\{f(t)\} = F(s)$is the Laplace transform of $f(t)$ with the parameter s[10], it is defined as

$$F(s) = L\{f(t)\} = \int_0^\infty f(t) e^{-st}\, dt \qquad (4.18)$$

We can find $L\{t^{\beta-1}\}$ from (4.18)

$$L\{t^{\beta-1}\} = \int_0^\infty t^{\beta-1} e^{-st}\, dt = \frac{1}{s^\beta}\Gamma(\beta) \qquad (4.19)$$

Laplace transform has the following Property

$$L\{f(t)e^{at}\} = F(s-a) \qquad (4.20)$$

Then we can write $L\left\{t^{\beta-1} e^{\frac{-t}{\delta}}\right\}$ as

$$L\left\{t^{\beta-1} e^{\frac{-t}{\delta}}\right\} = \frac{\Gamma(\beta)}{\left(s + \frac{1}{\delta}\right)^\beta} \qquad (4.21)$$

When we substitute (4.21) in (4.17), we get

$$g(-s) = \frac{1}{\delta^{\beta}\Gamma(\beta)} L\{f(t)\} \frac{\Gamma(\beta)}{\left(s+\frac{1}{\delta}\right)^{\beta}}$$

$$= \frac{1}{\delta^{\beta}} F(s) \frac{1}{\left(s+\frac{1}{\delta}\right)^{\beta}} = \frac{1}{\delta^{\beta}} F(s) \frac{1}{\frac{1}{\delta^{\beta}}(\delta s+1)^{\beta}}$$

$$= F(s) \frac{1}{(\delta s+1)^{\beta}} \quad (4.22)$$

Let$t = -s$, we can rewrite the moment generating function $g(t)$ as

$$g(t) = F(-t) \frac{1}{(1-\delta t)^{\beta}} \quad (4.23)$$

For $t \leq 0$

Let us substitute $t = 0$, we get

$$g(0) = E(e^{0}) = E(1) = \int_{0}^{\infty} h(x)\ dx = F(0) \frac{1}{\left(1-\delta(0)\right)^{\beta}} \quad (4.24)$$

From Laplace transform

$$F(0) = \int_{0}^{\infty} f(t) e^{-(0)t}\ dt = \int_{0}^{\infty} f(t)\ dt = 1 \quad (4.25)$$

Because $f(t)$ is a density function, we can substitute (4.25) in (4.24), we get

$$\int_{0}^{\infty} h(x)\ dx = (1) \frac{1}{\left(1-\delta(0)\right)^{\beta}} = 1 \times 1 = 1 \quad (4.26)$$

That means $h(x)$ is a density function.

Conclusion:

In this paper we presents the basic properties of Mittag-Leffler density, and use convolution theorem to study the moment generating function of sum of two random variables, this theorem allow us to work with highest order of moment generating function of Mittag-Leffler density.

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